Directing nonlinear dynamic systems to any desired orbit

Liu Zonghua
CCAST (World Laboratory), Beijing 8730, China;
Graduate School, China Academy of Engineering Physics, P.O. Box 8009, Beijing, 100088, People's Republic of China;
and Department of Physics, Guangxi University, Nanning, 530004, People's Republic of China

Chen Shigang
Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, People's Republic of China
(Received 16 July 1996)

A method of controlling arbitrary nonlinear dynamic systems, $dx/dt=F(x,t)$ $(x\in R^n)$, is presented. It is proved that the system can be entrained to any arbitrary ''goal'' dynamics $g(t)$ by use of the open-loop action and adjusting the control parameter at the same time. Examples of some entrainment ''goals'' are given for the Lorenz and the Rössler systems. The basins of entrainment are also established for the Lorenz, Rössler, and Duffing systems. Numerical studies show that this method works as well as the closed-loop control method [Physica D **85**, 1 (1995); Phys. Lett. A **213**, 148 (1996)]. [S1063-651X(97)04701-6]

PACS number(s): 05.45.+b, 03.20.+i, 46.10.+z

I. INTRODUCTION

The open-loop control method was first introduced by Hübler [1] and used by Hübler and Lüscher [2,3] in studies involving the logistic map and nonlinear damped oscillators. It was then developed towards complex dynamics with many dynamic attractors by Jackson [4–6]. Its aim is to achieve the desired global dynamics for dynamical systems that are not necessarily chaotic, provided that some restrictions on the initial conditions that stand for the values of the state variables at the time when control is initiated and on the target dynamics are met. Though the control (perturbation) may be large and a system model is required, obtaining any dynamics by this approach is attractive compared to the only unstable periodic orbits stabilized by the Ott, Grebogi, and Yorke method [7], by a delayed feedback approach [8], or by a linear feedback control [9], and compared to a series of new dynamics available by a self-interaction method [10], in which, although many dynamics are possible, the resulting dynamics cannot be arbitrary.

Many of the important dynamic models consist of systems of ordinary differential equations,

$$\dot{x}=F(x,t) \quad (x\in R^n). \tag{1}$$

If one wants to entrain the solution of the system to some ''goal'' behavior $g(t)$ in order to obtain

$$\lim_{t\to\infty}[x(t)-g(t)]=0 \tag{2}$$

assuming that the dynamics is of the form

$$\dot{x}=F(x,t)+K(g,t)S(t) \tag{3}$$

it follows that $K(g,t)$ must satisfy [4]

$$K(g,t)=\dot{g}-F(g,t), \tag{4}$$

where $S(t)$ is a ''switching function'', $S(t)=0$ $(t<t_0)$ and e.g., $S(t)=1$ $(t\geqslant t_0)$. The global region of phase space in which the initial condition $x(0)$ yields solutions of Eq. (3) satisfying Eq. (2) is referred to as the basin of entrainment BE(g). The control equation (3) is only initiated when $x(0)\in$BE(g) (which defines $t=t_0$) after which the state of the system does not need to be monitored to ensure control. Therefore, $g(t)$ cannot be implemented freely. These limitations then deteriorate this attractive controlling method.

In order to remove the limitations, Jackson and Grosu [11] present an approach that can entrain dynamical systems to any desired orbits (goal dynamics) by using the open-plus-closed-loop (OPCL) control method, that is

$$K(g,t)=\dot{g}-F(g,t)+C(g,t)[g(t)-x(t)], \tag{5}$$

where $C(g,t)[g(t)-x(t)]$ is a linear closed-loop control. Chen [12] gives out some numerical studies of the OPCL control method. In this paper, we present a method of controlling arbitrary nonlinear dynamics systems. It is proved that this method can implement the entrainment of arbitrary goal dynamics $g(t)$ by using the open-loop action and adjusting the control parameter, the open-plus-adjusting parameter (OPAP), at the same time. The effect of this OPAP method is as effective as the OPCL control method [11,12]. Numerical studies show that the OPAP method can implement the entrainment from one system to another system and the entrainment in which the goal dynamics is a high-dimensional torus. The basins of entrainment are also established for the Lorenz, Rössler, and Duffing systems.

II. THE OPEN-PLUS-ADJUSTING PARAMETER (OPAP) CONTROL METHOD

If we adjust the parameter of Eq. (1) as

$$p(t)=\overline{p}-\epsilon[x(t)-g(t)] \tag{6}$$

then Eq. (3) becomes

$$\dot{x}=F(x(t),p(t))+S(t)[\dot{g}-F(g,\overline{p})], \tag{7}$$

where $\overline{p}$ is the nominal value of parameter $p(t)$ and ϵ is the control variable. Let

$$u(t)=x(t)-g(t) \tag{8}$$

then we have

$$\frac{du(t)}{dt}=F(g(t)+u(t),p(t))-F(g(t),\overline{p}). \tag{9}$$

$u(t)$ is asymptotically stable if all the eigenvalues of the Jacobian matrix of Eq. (9) have negative real parts. Because the goal dynamics $g(t)$ needs to be located in the convergent region of the original dynamics, we can utilize the control variable ϵ to adjust the convergent region such that the selected $g(t)$ is guaranteed in the modified convergent region. For the three-dimensional system, the Jacobian matrix of Eq. (9) is $J=(a_{ij})$. By Routh-Hurwitz stability criterion, introduced by Jackson [4], all the eigenvalues of the Jacobian matrix J have negative real parts if three inequality equations

$$\begin{aligned} &a_1>0,\\ &a_3>0,\\ &a_1a_2-a_3>0 \end{aligned} \tag{10}$$

are satisfied, where a_1, a_2, and a_3 are coefficients of the characteristic polynomial $\lambda^3+a_1\lambda^2+a_2\lambda+a_3$, determined from the Jacobian matrix J. Now we use the Lorenz, Rössler, and Duffing systems as examples to show that arbitrary dynamics can be obtained by the control inputs proposed in Eq. (7).

For the Rössler system

$$\begin{aligned} \dot{x}&=-y-z,\\ \dot{y}&=x+ay,\\ \dot{z}&=b+z(x-c), \end{aligned} \tag{11}$$

we take the goal dynamics to be

$$g(t)=(g_x,g_y,g_z)=(\sin t,2+\cos\lambda^{-1}t,\sin\lambda^{-2}t), \tag{12}$$

where the parameter $a=b=0.2$, $c=4.5$, and the frequency $\lambda=1.839\,286\,755\,21$, which is the real root of the equation $\lambda^3-\lambda^2-\lambda-1=0$. Because λ is an irrational number, the ratios of frequency between the three components of $g(t)$ are also irrational. This is different from the examples given in Ref. [12] in which the ratios are all rational. Now we adjust the parameter a as follows:

$$a(t)=\overline{a}-\epsilon[y(t)-g_y]; \tag{13}$$

the control equation (7) becomes

$$\begin{aligned} \dot{x}&=-y-z+S(t)(\dot{g}_x+g_y+g_z),\\ \dot{y}&=x+a(t)y+S(t)(\dot{g}_y-g_x-\overline{a}g_y),\\ \dot{z}&=b+z(x-c)+S(t)[\dot{g}_z-b-g_z(g_x-c)]. \end{aligned} \tag{14}$$

The Jacobian matrix corresponding to Eq. (9) is

$$J=\begin{pmatrix} 0 & -1 & -1\\ 1 & \overline{a}-\epsilon g_y & 0\\ g_z & 0 & g_x-c \end{pmatrix}. \tag{15}$$

According to Eq. (10), we let

$$\epsilon=\frac{1+\overline{a}}{g_y}. \tag{16}$$

For avoiding the variation of parameters too large, we let $a(t)=\overline{a}$ when $|a(t)|>5.0$, that is, we let the system run freely at this time. The control result is as in Fig. 1, where $t\leqslant 11$ denotes the free motion of the system, $11<t\leqslant 20$ the effect of adding open-loop action $K(g,t)$, and $t>20$ the effect of the OPAP control method. From (a), (b), and (c) of Fig. 1, we can see that it cannot be entrained to goal dynamics by adding only open-loop control. The entrainment can be implemented only after adding open-loop action and adjusting parameters at the same time. From (d) and (e) of Fig. 1, we can see that, because the frequency is irrational, the trajectory fills the region of goal dynamics.

In the second example we consider the Lorenz model

$$\begin{aligned} \dot{x}&=\sigma(y-x),\\ \dot{y}&=\gamma x-y-xz,\\ \dot{z}&=xy-bz, \end{aligned} \tag{17}$$

where $\sigma=10$, $\gamma=28$, and $b=\frac{8}{3}$. In order to illustrate the arbitrariness of goal dynamics, we take the Rössler system as our goal dynamics, where the parameters of the Rössler system $a=b=0.2$ and $c=5.7$. Because there are interactions among the variables of goal dynamics x, y, and z this goal dynamics is much more difficult than that of Refs. [11,12]. Now we adjust two parameters of the Lorenz model, σ, γ,

$$\begin{aligned} \sigma(t)&=\overline{\sigma}-\epsilon_1[x(t)-g_x],\\ \gamma(t)&=\overline{\gamma}-\epsilon_2[y(t)-g_y], \end{aligned} \tag{18}$$

where $\overline{\sigma}$ and $\overline{\gamma}$ are the nominal values of parameters σ and γ respectively, g_x and g_y are the variables of the Rössler system x and y. According to Eq. (10), we let

$$\begin{aligned} \epsilon_1&=\begin{cases} \dfrac{23}{(g_y-g_x)} & \text{if } |g_y-g_x|>0.01\\ 0 & \text{otherwise,} \end{cases}\\ \epsilon_2&=\begin{cases} \dfrac{9}{g_x} & \text{if } |g_x|>0.01\\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{19}$$

In order to avoid the too large a variation of parameters, we let $\sigma(t)=10$ when $|\sigma|>20$ and $\gamma(t)=28$ when $|\gamma|>50$. Figure 2 is the control result where $t\leqslant 20$ denotes the free motion of the system, $20<t\leqslant 40$ the effect of adding open-loop action $K(g,t)$, $t>40$ the effect of the OPAP control method. (d) and (e) of Fig. 2 denote the control results at different control initial values; we can see that they all go to the attractor of the Rössler system, so the OPAP control method is independent of the ‘‘switching on’’ time. (f) of Fig. 2 denotes the

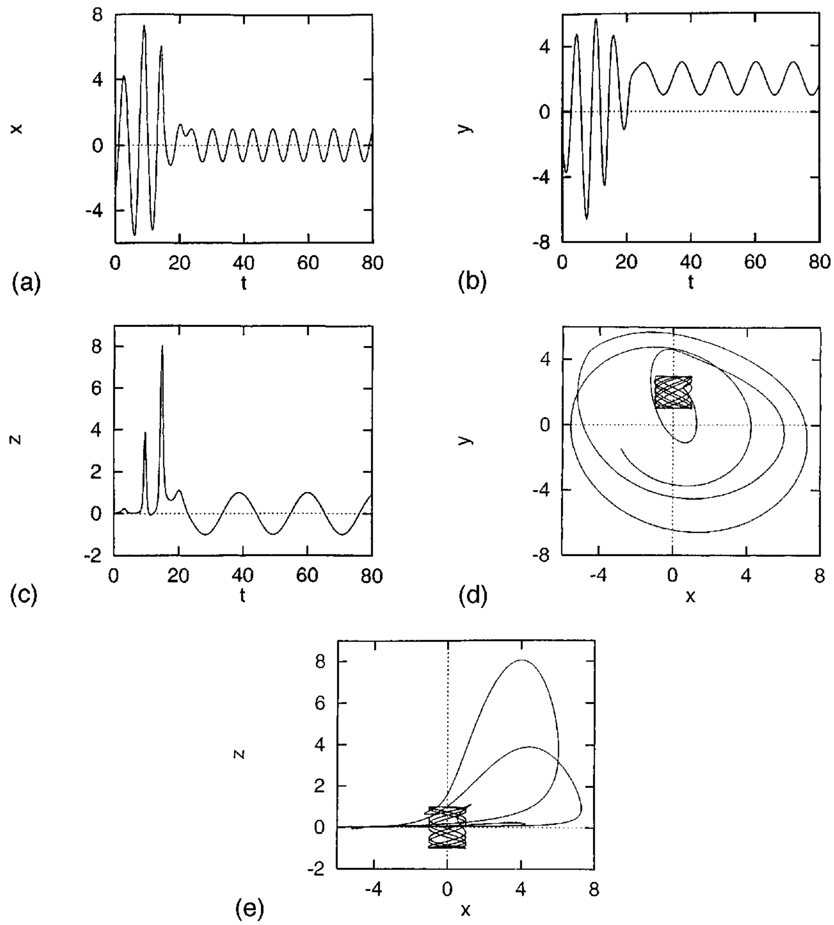

FIG. 1. Response of a controlled Rössler system with goal dynamics $(\sin t,\ 2+\cos\lambda^{-1}t,\ \sin\lambda^{-2}t)$ where $\lambda=1.839\,286\,755\,21$. $t\leqslant 11$ denotes the free motion of the system; $11<t\leqslant 20$ the effect of adding open-loop action; $t>20$ the effect of the OPAP method. (a) $x(t)\sim t$; (b) $y(t)\sim t$; (c) $z(t)\sim t$; (d) the asymptotic dynamics in (x,y) plane; (e) the asymptotic dynamics in (x,z) plane.

control process in phase space (x,y). It is clear that first the system locates at the Lorenz attractor, then it oscillates around the Lorenz attractor, and finally it rests on the Rössler attractor.

In the last example we use the Duffing equations in the form

$$\dot{x}=y,$$
$$\dot{y}=-\gamma y-\alpha x-x^3, \tag{20}$$

and take the goal dynamics to be $g_x=B\sin(\omega t)$, $g_y=B\omega\cos(\omega t)$, where $\alpha=1$, $\gamma=0.1$, and $B=\omega=3$. This is a nonchaotic system. Making the similar analysis with the above two examples, we let

$$\alpha(t)=\overline{\alpha}-\epsilon_1[x(t)-g_x],$$
$$\gamma(t)=\overline{\gamma}-\epsilon_2[y(t)-g_y], \tag{21}$$

$$\epsilon_1=\begin{cases}\dfrac{-3+3g_x^2}{g_x} & \text{if } |g_x|>0.01\\ 0 & \text{otherwise,}\end{cases}$$
$$\epsilon_2=\begin{cases}\dfrac{\overline{\gamma}-4.2}{g_y} & \text{if } |g_y|>0.01\\ 0 & \text{otherwise}\end{cases} \tag{22}$$

and let $\alpha(t)=1$ when $|\alpha(t)|>10$, and $\gamma(t)=0.1$ when $|\gamma(t)|>5.0$. Figure 3 is our control result. Obviously, the system is entrained to goal dynamics.

III. BASINS OF ENTRAINMENT

In this section we explore the problem of determining the size of the basin of entrainment $\mathrm{BE}(g|t_0)$. To study this problem, we expand $F(g+u,p(t))$ to obtain the nonlinear generalization of Eq. (9)

$$\frac{du_i}{dt}=\frac{\partial F_i}{\partial g_j}u_j+\frac{\partial F_i}{\partial p_j}(-\epsilon u_j)+\frac{1}{2}\frac{\partial^2 F_i}{\partial g_j\partial g_k}u_ju_k+\frac{1}{3!}\frac{\partial^3 F_i}{\partial g_j\partial g_k\partial g_l}u_ju_ku_l+\cdots. \tag{23}$$

In all examples considered in this study $F(x,t)$ is at most a third-degree polynomial, so there are no additional terms to Eq. (23).

For the Lorenz model Eq. (17),

$$\dot{u}_1=-[\sigma+(g_y-g_x)\epsilon_1]u_1+\sigma u_2,$$
$$\dot{u}_2=(\gamma-g_z)u_1-(1+g_x\epsilon_2)u_2-g_xu_3-u_1u_3,$$
$$\dot{u}_3=g_yu_1+g_xu_2-bu_3+u_1u_2. \tag{24}$$

In order to get the Lyapunov function for Eq. (24), we study

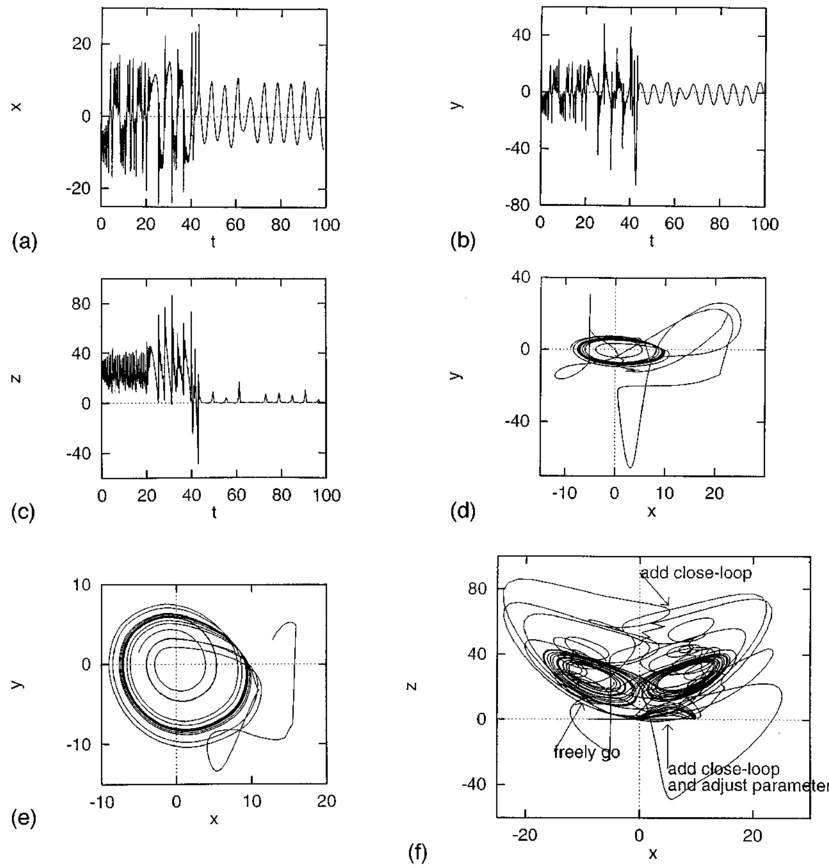

FIG. 2. The entrainment from the Lorenz model to the Rössler system. $t\leqslant 20$ denotes the free motion of the system; $20<t\leqslant 40$ the effect of adding open-loop action; and $t>40$ the effect of OPAP method. (a) $x(t)\sim t$; (b) $y(t)\sim t$; (c) $z(t)\sim t$; (d) and (e) the effect of the OPAP method at different initial points; (f) the asymptotic dynamics in (x,z) plane.

$$\frac{1}{2}\frac{d}{dt}(\rho u_1^2+u_2^2+u_3^2)=-(1+g_x\epsilon_2)(\alpha u_1-u_2)^2 - b(\beta u_1-u_3)^2 \tag{25}$$

provided that $\dot\rho=0$ and (ρ,α,β) satisfy

$$2\alpha(1+g_x\epsilon_2)=\sigma\rho+\gamma-g_z,$$
$$2\beta b=g_y, \tag{26}$$
$$(1+g_x\epsilon_2)\alpha^2+b\beta^2=\rho[\sigma+(g_y-g_x)\epsilon_1].$$

One finds that these conditions require that

$$\beta=\frac{g_y}{2b},$$

$$\alpha=1+(g_y-g_x)\frac{\epsilon_1}{\sigma}\pm\left(\left[1+(g_y-g_x)\frac{\epsilon_1}{\sigma}\right]^2 - \frac{b\beta^2+(\gamma-g_z)\left[1+(g_y-g_x)\frac{\epsilon_1}{\sigma}\right]}{1+g_x\epsilon_2}\right)^{1/2}, \tag{27}$$

$$\rho=\frac{(1+g_x\epsilon_2)\alpha^2+b\beta^2}{\sigma+(g_y-g_x)\epsilon_1}.$$

If we select ϵ_1 and ϵ_2 so that $1+g_x\epsilon_2>0$ and $\sigma+(g_y-g_x)\epsilon_1>0$, we have $\rho>0$ and Eq. (25) is less than 0. For real α, and $\dot\rho=0$,

$$\left[1+(g_y-g_x)\frac{\epsilon_1}{\sigma}\right]^2-\frac{\frac{g_y^2}{4b}+(\gamma-g_z)\left[1+(g_y-g_x)\frac{\epsilon_1}{\sigma}\right]}{1+g_x\epsilon_2}>0, \tag{28}$$

$$\beta\dot g_y-\alpha^2 g_x\epsilon_2=\sigma\alpha\dot g_z+\rho\epsilon_1(\dot g_y-\dot g_x).$$

Therefore we have the following result: $\rho u_1^2+u_2^2+u_3^2$ can be used as the Lyapunov function and ensures global entrainment provided that Eqs. (27) and (28) are satisfied.

For the Rössler system, we implement the goal entrainment in the first example by only adjusting parameter a. In order to discuss the basin of entrainment, now we consider a general situation and adjust the parameters of system a, b, and c:

$$a(t)=\overline{a}-\epsilon_1[y(t)-g_y],$$
$$b(t)=\overline{b}-\epsilon_2[x(t)-g_x], \tag{29}$$
$$c(t)=\overline{c}-\epsilon_3[z(t)-g_z].$$

We have

$$\dot u_1=-u_2-u_3,$$
$$\dot u_2=u_1+(a-\epsilon_1 g_y)u_2, \tag{30}$$
$$\dot u_3=(g_z-\epsilon_2)u_1+(g_x-c+g_z\epsilon_3)u_3+u_1u_3.$$

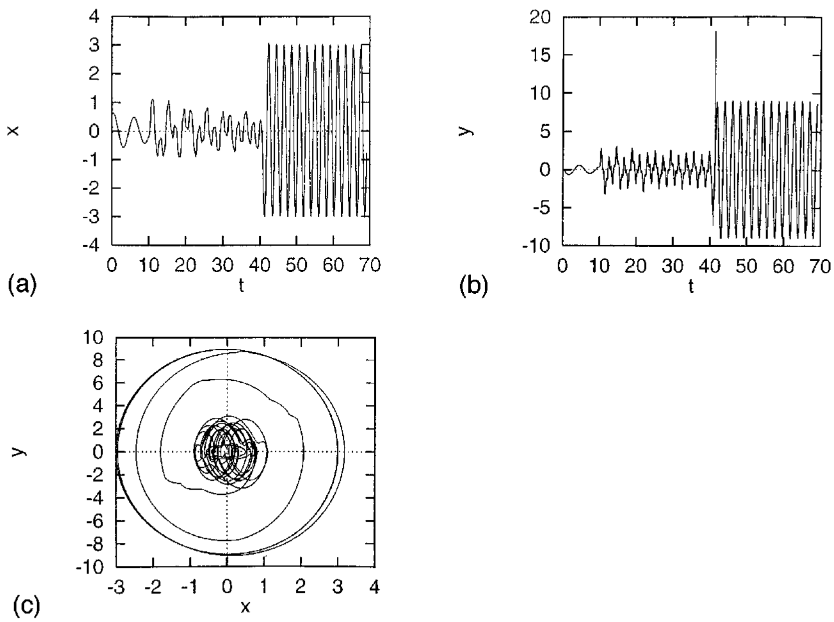

FIG. 3. The Duffing equation is used to illustrate the entrainment of the nonchaotic system, where the goal dynamics is $(3\sin 3t,\ 9\cos 3t)$. $t\leqslant 10$ denotes the free motion of the system; $10<t\leqslant 40$ the effect of adding open-loop action; $t>40$ the effect of the OPAP method. (a) $x(t)\sim t$; (b) $y(t)\sim t$; (c) the asymptotic dynamics in phase space (x,y).

It will produce a simple Lyapunov function

$$\frac{1}{2}\frac{d}{dt}(u_1^2+u_2^2+u_3^2)=(g_z-\epsilon_2-1)u_1u_3-(\epsilon_1 g_y-a)u_2^2 -(c-g_x-u_1-\epsilon_3 g_z)u_3^2. \tag{31}$$

If we let ϵ_1, ϵ_2, and ϵ_3 satisfy $\epsilon_2=g_z-1$, $\epsilon_1 g_y-a>0$, and $c-g_x-u_1-\epsilon_3 g_z>0$, then $\frac{1}{2}(d/dt)(u_1^2+u_2^2+u_3^2)<0$. So the basin of entrainment of the Rössler system is global. In our first example we only adjust parameter a, so it is not global. Why can we implement the entrainment of goal dynamics? That is because we can select the control time of open-loop action until the system enters the convergent region where the eigenvalues of Eq. (15) are negative. Then we add the OPAP method so it can let the system go to the goal dynamics.

For the Duffing equation, we have

$$\begin{aligned}\dot{u}_1&=u_2,\\ \dot{u}_2&=(\epsilon_1 g_x-1-3g_x^2)u_1-(\gamma-\epsilon_2 g_y)u_2-3g_x u_1^2-u_1^3.\end{aligned} \tag{32}$$

From Ref. [13] we know that Eq. (32) can be written as the form $\ddot{u}+\gamma\dot{u}+dV(u,c)/du=F(t)$, where $V(u,c)$ is a nonlinear potential function

$$\frac{dV}{du}=(1+3g_x^2-\epsilon_1 g_x)u+3g_x u^2+u^3. \tag{33}$$

By $dV/du=0$, we get three extreme points

$$u_1=0,$$

$$u_{2,3}=-\tfrac{3}{2}g_x\pm\sqrt{\epsilon_1 g_x-1-\tfrac{3}{4}g_x^2}. \tag{34}$$

If we select a suitable ϵ_1 so that

$$\epsilon_1 g_x-1-\tfrac{3}{4}g_x^2<0, \tag{35}$$

$u_1=0$ is only an extreme point and

$$\frac{d^2V}{du^2}=1+3g_x^2-\epsilon_1 g_x>0. \tag{36}$$

So $u_1=0$ is a minimum point. We can let the system go to the attractor of $u_1=0$ at any point of phase space and implement the entrainment of goal dynamics. In fact, our last example is just a special case in which Eq. (22) satisfies Eq. (35).

IV. CONCLUSION

In this study we have shown that the OPAP control method works as well as the OPCL control method. It can implement the entrainment of goal dynamics which cannot be implemented by only open-loop action. In our control process, we first switch on the open-loop control $[S(t_0)=1]$ and find that it cannot implement the entrainment of goal dynamics. Then we adjust the control parameters of the system to some suitable values so that the system is entrained to goal dynamics. The example of the Rössler system illustrates that the OPAP control method can implement the entrainment of goal dynamics whose ratios of frequency are irrational and the example of the Lorenz model illustrates that the OPAP control method can implement the entrainment between different attractors of different systems. In Sec. III the extent of the basins of entrainment for the Rössler and Duffing systems is shown to be global (the entire phase space) provided the control parameters be selected suitably. For the Lorenz model, the basin of entrainment is global only when Eqs. (27) and (28) are satisfied.

ACKNOWLEDGMENT

This work was supported partially by the National Natural Science Foundation of China.

[1] A. W. Hübler, Dissertation, Department of Physics, Technical University of Munich, 1987 (unpublished).
[2] A. Hübler and E. Lüscher, Naturwissenschaften **76**, 67 (1989).
[3] E. Lüscher and A. Hübler, Helv. Phys. Acta **62**, 543 (1989).
[4] E. A. Jackson, Phys. Rev. A **44**, 4839 (1991).
[5] E. A. Jackson, Physica D **50**, 341 (1991).
[6] E. A. Jackson, Phys. Lett. A **151**, 478 (1990).
[7] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[8] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[9] B. A. Huberman and E. Lumer, IEEE Trans. Circuits Syst. **37**, 547 (1990).
[10] C.-C. Chen, C.-H. Tsai, and C.-C. Fu, Phys. Lett. A **194**, 265 (1994).
[11] E. A. Jackson and I. Grosu, Physica D **85**, 1 (1995).
[12] C.-C. Chen, Phys. Lett. A **213**, 148 (1996).
[13] K. Chang, A. Kodogeorgiou, A. Hübler, and E. A. Jackson, Physica D **51**, 99 (1991).